Bridging Units: Resource Pocket 4

Iterative methods for solving equations numerically

This pocket introduces the concepts of using iterative methods to solve equations numerically in cases where an algebraic approach is not possible, or too difficult for this level of mathematics. In the current 2007 Key Stage 3 Programme of Study there is a requirement for ‘numerical methods for solving equations’ and trial and improvement is specifically mentioned. The general use of recursive or iterative methods is not referred to.

Students progressing to the Higher tier would benefit from being aware of the idea of a numerical approach to solving equations and having a basic awareness of an iterative formula. They may already be familiar with the idea of recursive notation if they have studied Pocket 6 (Sequences) prior to this Pocket; however this is not essential as the notation is introduced here too.

Students will already be familiar with the concept of estimation when solving problems and will be able to substitute values into formulae and expressions. It is also important that they have some understanding of rearranging formulae in order to understand the derivation of the iterative formulae. Basic flow charts will be employed to show the individual steps of the process, as in the GCSE.

The notation used in this pocket will be the same as that used in the new GCSE specification. The content of this Pocket is not a requirement for student progressing to the Foundation tier, but Developing Understanding 1 and Skills Builder 1 would be useful consolidation of estimation.

This resource pocket progresses through three sections: developing understanding, skills builders and problem solving activities. As with all 9 resource pockets there are a number of different learning styles and approaches used to cater for a variety of learners.

1. Developing Understanding

   These are class based, teacher led questions with suggested commentary to get the most from a class or small group discussion. The boxed text can either be copied onto the whiteboard for class discussion, or printed onto cards and handed out to students to be used for paired or small group work.

2. Skills Builders

   These are standard progressive worksheets that can be used to drill core skills in a particular area. Skills Builder 2 includes three questions on trial and improvement – at this stage making a decision about the final solution can be done by simply choosing the nearest value, although Developing Understanding 2 goes through the checking of the midpoint between the two nearest solution values for more able students.

3. Problem Solving Activities

   Extension activities for paired work or small group work to develop problem solving skills whilst focussing on a particular area of mathematics that students can learn to apply.

Developing Understanding 1

Charlie chooses a number and types it into his calculator

When he squares the number and then doubles the answer, he gets 400.

What is Charlie's number?

Ask students

- Could we express this problem in algebra?
- How might we go about finding the number?
- Will the answer be an integer? How do you know?
- How can we write our answers?

Give students some time to work with their calculators to try to work out a relatively accurate answer. It may be that some students correctly deduce that the answer is $\sqrt{200}$ and use the square root button on their calculator to deduce the answer 14.142135… Others may use a 'trial and error' approach.

Discuss with students the various approaches they have used. Draw out elements that show a systematic strategy e.g. noting whether guesses gave answers that were too high/too low, and choosing the next value accordingly.

Explain that there are some equations which are difficult to solve using algebra and so these 'numerical' approaches are needed. State that to start this process, practice is needed at estimating roughly the correct size of the answer. In the case of the question above, the answer is between 14 and 15.

Now display this box on the whiteboard:

Find the two integer values that are closest to the solution to these problems:

1. A number is chosen and is then squared.
   Double the original number is added to give a total of 89
2. A number is chosen and is then cubed.
   The original number is then added to give a total of 500.
3. A number is chosen and is then squared.
   The original number is cubed and this is added to give a total of 1000.

Can you express these problems algebraically?

Students could work on these problems individually or in pairs. Mini whiteboards would be helpful for rough work.

Bring the group back together and discuss approaches, including how students formulated algebra to represent these problems. For more able students it would be worth discussing the fact that Charlie's problem was much easier as the algebra can be rearranged:

Square the number and double it to get an answer of 400 can be formulated as

$$2x^2 = 400$$
$$x^2 = 200$$
$$x = \sqrt{200}$$

Since we have a square root button on our calculators, this can be easily solved.

However, the algebra in each of the other cases cannot be easily solved – although students are unlikely to have studied quadratics and cubics at this stage, they should be able to see that the usual steps of rearranging a formula do not hold as the x term cannot be isolated on one side of the equation.

The answers are

	Wording	Algebra	Nearest integer values
1	A number is chosen and is then squared. Double the original number is added to give a total of 80	$x^2 + 2x = 89$	8 and 9
2	A number is chosen and is then cubed. The original number is then added to give a total of 500	$x^3 + x = 500$	7 and 8
3	a number is chosen and is then squared. The original number is cubed and this is added to give a total of 1000	$x^2 + x^3 = 1000$	9 and 10

Developing Understanding 2

This section starts by discussing how to increase the accuracy of an estimated solution, moving from integer approximations to approximations to one decimal place. The section then moves on to introduce the idea of using an iterative method using a simple flow chart structure.

Explain that we sometimes need a more accurate solution to a problem than the nearest integer, perhaps to 1 decimal place. In this section we will work on getting more accurate solutions to problems, using formulae that we are given. Where the formulae are derived from follows in the next section.

Display the following box on the board:

What is the square root of 70 to 1 decimal [place?

How did you decide which values to trial?

How did you make your final decision about which value to choose?

The answers are

Trial value	Square of the trial value	Comment
8	64	
9	81	
8.5	72.25	Too high
8.4	70.56	Too high
8.3	68.89	Too low

8.4 is nearest and so students are likely to choose this value

Students might explain decisions such as:

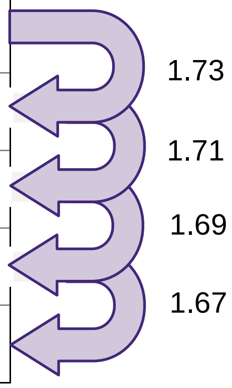
- Find the integer values closest to then solution first;
- Try the halfway point between the integer values;
- If a value is too low, try a higher one; if it is too high try a lower one;
- Choose the value that gives a value nearest to 70.

The final point is not quite true. This would only hold if the function was linear – in this case the function is quadratic and so does not increase at a constant rate.

Ask students to work out the differences between some subsequent values to illustrate this:

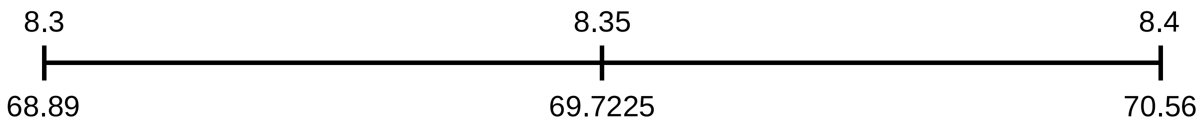
Ask students to work out the differences between some subsequent values to illustrate this:

Trial value	Square of the trial value
8.7	75.69
8.6	73.96
8.5	72.25
8.4	70.56
8.3	68.89



Explain that this method is called 'trial and improvement' and will be covered in more detail during the GCSE course. There are some simple examples in questions 1 - 3 of Skills Builder 2.

If you wish, explain how we choose the correct value to 1 decimal place, i.e. choose the two nearest values to 1 decimal place and then check the midpoint of these two values to determine the correct solution. A number line as follows can help:



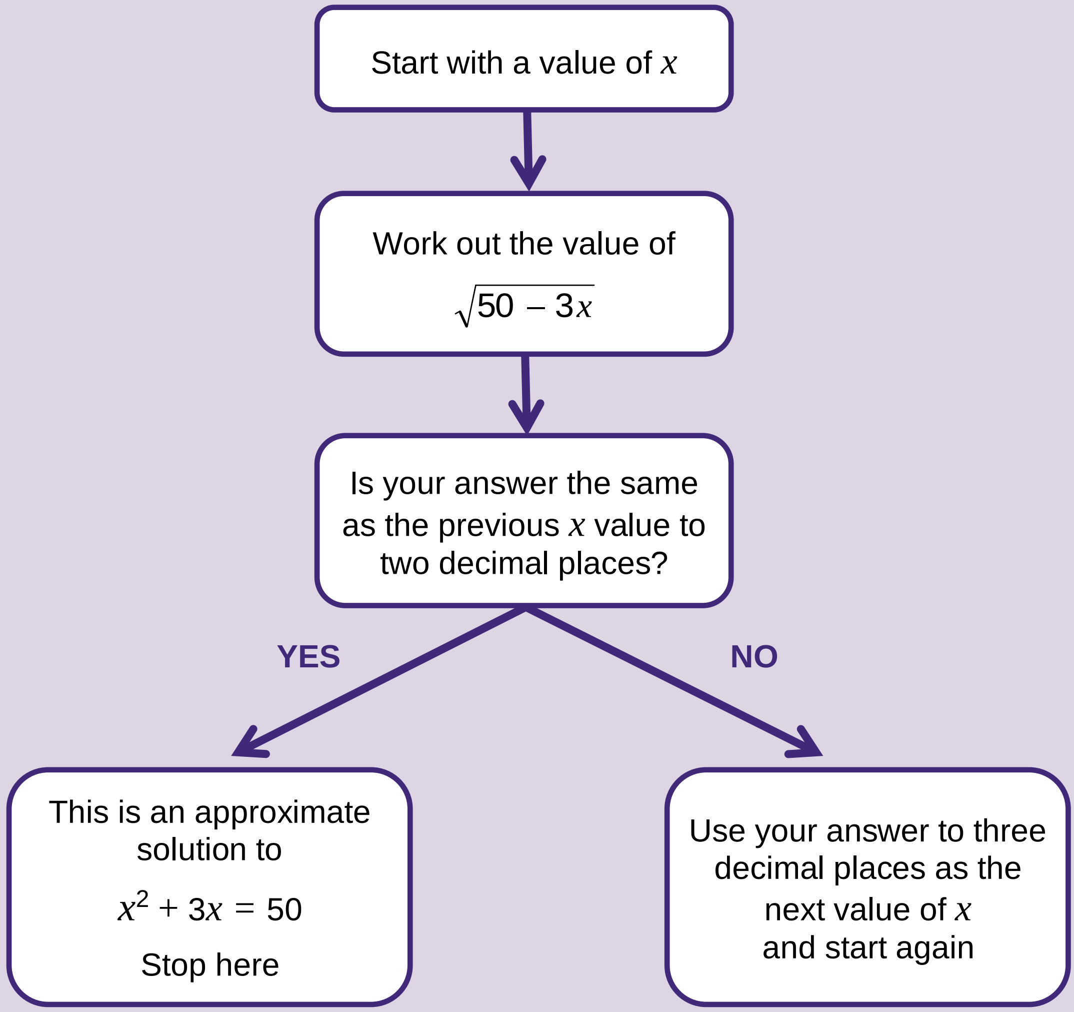
Here, the correct solution cannot be in the range 8.3 - 8.35 as the squares of these values are all below 70; hence the correct solution must be between 8.35 and 8.4 and all values in this range round to 8.4 to one decimal place and so this is the correct solution.

Explain we will now move on to a different method for finding a numerical solution to a problem. Whilst this is an alternative method, it will come to the same solutions as we found earlier. This method will use a flow chart to explain the steps to follow, and these steps are often repeated several times. Repeat the steps until the final condition is met.

Display the following box on the board:

If time allows, allow students to choose any value as a starting value – some of these will work well and converge to a solution; others will diverge and move progressively further away from the solutions. If time is more limited, ask half the class to start with $x = 4$ and the other half with $x = 0$

This flow chart will find a solution to the equation $x^2 + 3x = 50$



Starting with $x = 4$
4
6.164
5.613
5.759
5.720
5.731
5.728
5.728 and 5.731 both round to 5.73 to 2 d.p

Starting with $x = 0$
0
7.071
5.365
5.823
5.704
5.735
5.727
5.729
5.727 and 5.729 both round to 5.73 to 2 d.p.

Ask students:

- How many iterations did you need?
- How can we check if the answer is correct?
- Might there be any other solutions?
- Did anyone's starting value not lead to the solution 5.73? What happened?

Starting with x = 4 requires 6 iterations; starting with x = 0 requires 7 iterations. Note that even starting much further away from then solution only increased the number of iterations by one in this case. Mention the word 'convergence' for the way that the iterative method often ensures convergence to the solution to the equation.

We can check if the answer is correct by substituting it back into $x^2 + 3x$ and checking if we get the answer of 50. (The answer is actually 50.0229 so it is very close). Later in their study of mathematics, students will be able to check the answer using the quadratic formula.

Students might suggest that there is another solution by taking the negative square root rather than the positive square root. If time allows, suggest a starting value of x = –10 and ask students to try to find the negative root.

Starting with x = – 10
–10
–8.944
–8.765
–8.735
–8.730
–8.729
–8.730 and –8.729 both round to –8.73 to 2 d.p.

If a student originally started with a value of their own choosing that was 17 or greater they would have experienced a situation where they tried to take the square root of a negative number. This problem would also occur if they started with a value such as x = –100 as the first iteration gives 18.708 and the same problem would occur on the second iteration.

Skills Builder 2, from question 4 onwards, provides practice in the use of these flow charts. It will be helpful for these questions to show students how to use the memory facility or the 'ANS' facility to save typing in long numbers repeatedly.

Developing Understanding 3

This section builds on Developing Understanding 2, starting with an introduction to the use of iterative notation. This first section of material is repeated in Pocket 6 (Number sequences), so if students have already completed Pocket 6 the majority of this page and the next could be omitted.

After the introduction of iterative notation, iterative formulae will be used to find the approximate solution to an equation. The origins of the iterative formula, through the rearrangement of the original equation, are also considered.

Display the following information on the board. Students could work in pairs or small groups to determine the answers.

Lindsay is generating some numbers using a different term-to-term rule for each sequence.

1 1, 2, 4, 8, ….

2 5, 10, 20, 40, ….

3 1000, 500, 250, 125, ….

4 19, 17, 15, 13, ….

5 2, 4, 6, 10, 16, ….

Work out the term to term rule for each sequence.

The answers are:

1 Double the previous term starting with 1

2 Double the previous term, starting with 5

3 Double the previous term and add 1, starting with 2

4 Half the previous term, starting with 1000

5 Subtract 2 from the previous term starting with 19

6 Add together the two previous terms, starting with 2 and 4 as the first two terms

Discuss with students how the wording here can be quite long-winded and so some mathematical terminology has been introduced to summarise the information.

Display this box:

u_1 represents the first term of the sequence

u_2 represents the second term of the sequence

u_3 represents the third term of the sequence

*

*

*

u_n represents the *n*th term of the sequence

Explain that we can write statements using these symbols for recursive sequences. They are generally of the form $u_{n+1} = f(u_n)$ ie $(n + 1)$th term is written as a formula in terms of the previous term.

(Do not express the recursive formulae on the board using function notation if students are not familiar with it or you think it would be off-putting).

Return to the first box showing Lindsay's sequences and display the worded answers alongside the sequences.

1	1, 2, 4, 8, ….	Double the previous term, starting with 1
2	5, 10, 20, 40, ….	Double the previous term, starting with 5
3	2, 5, 11, 23, ….	Double the previous term and add 1, starting with 2
4	1000, 500, 250, 125, ….	Half the previous term, starting with 1000
5	19, 17, 15, 13, ….	Subtract 2 from the previous term starting with 19
6	2, 4, 6, 10, 16, ….	Add together the two previous terms, starting with 2 and 4 as the first two terms

In relation to question 1, ask students:

- How could we write the sentence 'To get the next term, double the previous term' using mathematical notation.
- How could we indicate that the sequence should start at 1?

Lead students through a discussion of possible ways of writing the information, correcting any misconceptions. Deduce that we can express the sequence as $u_{n+1} = 2u_n$ with $u_1 = 1$.

Next, either support students in working through the notation for the remaining five questions, or allow them time to work through these themselves. Remind students that there will be two parts to the definition of the sequence - the recursive formula and the definition of the first term.

The answers are:

1 $u_{n+1} = 2u_n$ with $u_1 = 1$

2 $u_{n+1} = 2u_n$ with $u_1 = 5$

3 $u_{n+1} = 2u_n + 1$ with $u_1 = 2$

4 $u_{n+1} = \frac{u_n}{2}$ with $u_1 = 1000$

5 $u_{n+1} = u_n - 2$ with $u_1 = 19$

6 $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 2$, $u_2 = 4$

The last one of these is quite demanding – students might need support in working out how to write 'the term before the last term' and in noting that we need two terms in order to start the sequence.

In pocket 6, Developing Understanding 3, there is an additional activity to allow practice of the development of recursive notation, if needed at this stage.

Explain that now that students are familiar with recursive / iterative notation, the next phase is to tie this together with the material covered in 'Developing Understanding 2'. Tell students that rather than using a flow chart, a simple recursive/iterative formula will be provided which students will use to find the roots.

To simplify the process, students will be asked to stop when two iterations given the same answer to a certain accuracy (usually two or three decimal places).

Display the following on the board. Students could work in pairs or small groups to determine the answers. Mini whiteboards could be used to jot down the results of each iteration. It will be helpful if students can use the 'ANS' or memory facility on their calculator.

Alex is solving a problem.

The equation he needs to solve to find the solution to the problem is

$x^2 - 4x - 7 = 0$

One of the solutions can be obtained using the formula

$u_{n+1} = \sqrt{4u_n + 7}$

Starting with $u_1 = 0$, find the solution to Alex's problem to 2 d.p.

The iterations produce the following results

Value of u_n	Value of $u_{n+1} = \sqrt{4u_n + 7}$
0	2.645751311
2.645751311	4.193209421
4.193209421	4.875739706
4.875739706	5.148102449
5.148102449	5.25284778
5.25284778	5.292578872
5.292578872	5.307571525
5.307571525	5.313218055

The last two values round to 5.31 to 2 d.p.

Further practice at using iterative formulae is available in Skills Builder 3.

If students are comfortable with rearranging algebraic formulae, finish this section by asking students if they have spotted anything about how the iterative formulae are derived. The following box may help to prompt discussion - it shows the two formulae used in Developing Understanding 2 and 3 and the formulae used in Skills Builder 2, and the relevant equations they relate to.

Equation	Formula used to solve the equation
$x^2 + 3x = 50$	$\sqrt{50 - 3x}$
$x^3 - 5x + 2 = 0$	$\frac{x^2 + 2}{5}$
$2x^2 + 2x - 3 = 0$	$\frac{3}{2(x + 1)}$
$x^2 - 4x - 7 = 0$	$u_{n+1} = \sqrt{4u_n + 7}$

It is quite difficult to spot the method as, for the first three formulae, the '$x =$' at the beginning has not been included. Give students time to think about the relationship between the equation and the relevant iterative formula and if necessary drop a hint about rearranging the equation. Guide students to the conclusion that an iterative formula is obtained by rearranging the given equation into the form $x = f(x)$ and note that this can probably be done in a number of different ways for a given equation.

Display the following box on the board.

Anisa is solving a problem.

The find the solution she needs to solve the equation $x^3 - 2x^2 + 5x - 9 = 0$

Anisa wants to use an iterative method to solve the equation.

How many different rearrangements can you come up with that she could use?

Express your rearrangements using notation if possible.

Possible rearrangements and the corresponding iterative formula are

Rearrangement	Iterative formula
$x = \sqrt[3]{2x^2 - 5x + 9}$	$u_{n+1} = \sqrt[3]{2u_n^2 - 5u_n + 9}$
$x = \dfrac{2x^2 - 5x + 9}{x^2}$	$u_{n+1} = \dfrac{2u_n^2 - 5u_n + 9}{u_n^2}$
$x = \sqrt{\dfrac{x^3 + 5x - 9}{2}}$	$u_{n+1} = \sqrt{\dfrac{u_n^3 + 5u_n - 9}{2}}$
$x = \dfrac{x^3 + 5x - 9}{2x}$	$u_{n+1} = \dfrac{u_n^3 + 5u_n - 9}{2u_n}$
$x = 9 + 2x^2 - x^3$	$u_{n+1} = 9 + 2u_n^2 - u_n^3$

This activity is demanding and it is likely to be suitable only for the most able students.

Skills Builder 1: Formulating problems and finding approximate integer solutions

1 Nathan types a number into his calculator. He squares the number and multiplies it by 3 to get an answer of 800.

Formulate Nathan's problem using algebra.

Find the two positive integers which are closest to the solution to the problem.

2 Lily types a number into her calculator. She cubes the number and then divides the answer by 6 to get an answer of 40.

Formulate Lily's problem using algebra.

Find the two positive integers which are closest to the solution to the problem.

3 I type a number into my calculator. I add 3 to the number, square the answer and finally add on the original number.

(a) Which of the following equations represents this situation?

$x^2 + 3 + x$ $\quad$ $(x + 3)^2 + x$ $\quad$ $x^2 + 9 + x$ $\quad$ $9x^2 + x$

When I carry out these operations, my final answer is 450.

(b) Find the two positive integers which are closest to the solution to the problem.

4 Match up the information in the three columns below. For each problem find the corresponding algebraic equation and the two nearest integer values to the solution to the equation.

Problem	Equation	Nearest integers to the solution
Square the number and add the original number to get an answer of 500	$x^3 + 2x = 500$	19 and 20
Cube the number and add double the original number to get an answer of 500	$(x + 2)^2 + x = 500$	21 and 22
Square the number and add double the original number to get an answer of 500	$x^3 - 3x = 500$	21 and 22
Add two to the number and square it then add the original number to get an answer of 500	$x^2 + x = 500$	8 and 9
Cube the number and subtract three times the original number to get an answer of 500	$x^2 + 2x = 500$	7 and 8

Skills Builder 2: Basic trial and improvement and iterative flow charts

1 Without using the cube root button on your calculator, find the cube root of 100 to one decimal place.

2 Nadia types a number into her calculator, squares it and then adds the original number to get an answer of 60.

(i) Formulae Nadia's problem using algebra.

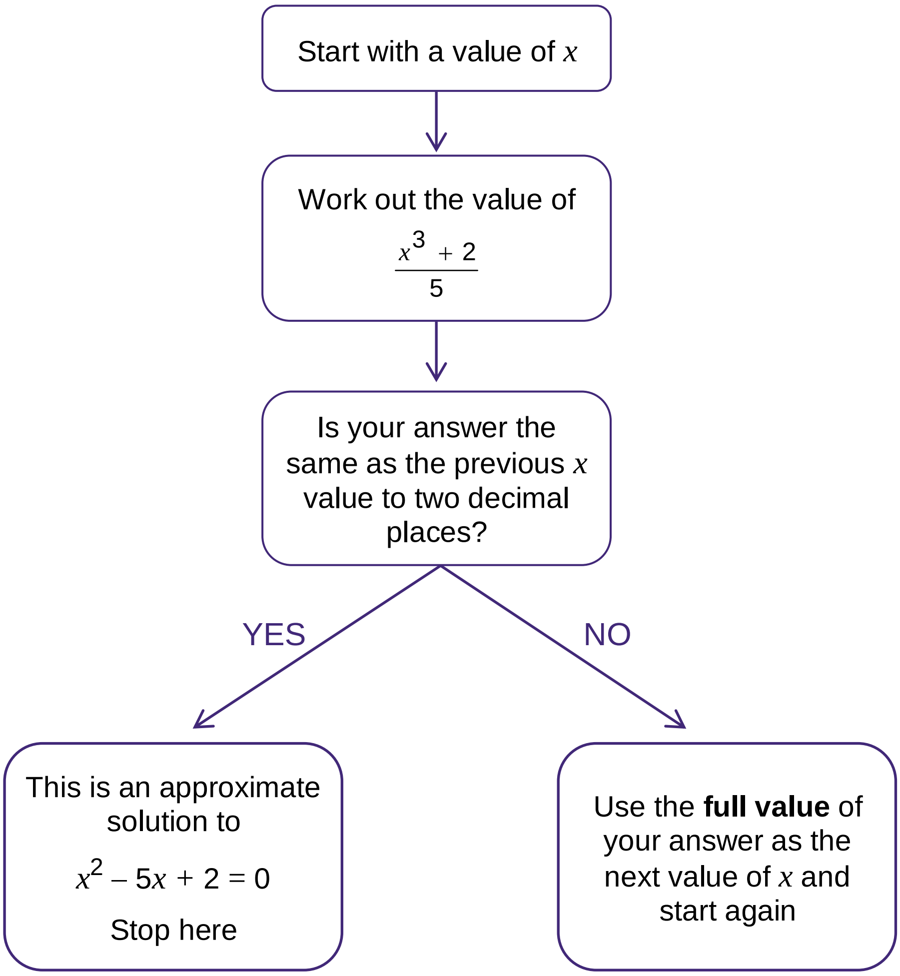
(ii) Find Nadia's number to one decimal place.

3 Scott types a number into his calculator, cubes it and then adds double the original number to get an answer of 100.

(i) Formulae Scott's problem using algebra.

(ii) Find Scott's number to one decimal place.

4 The flow chart is to be used to find solutions to the equation $x^3 - 5x - 2 = 0$



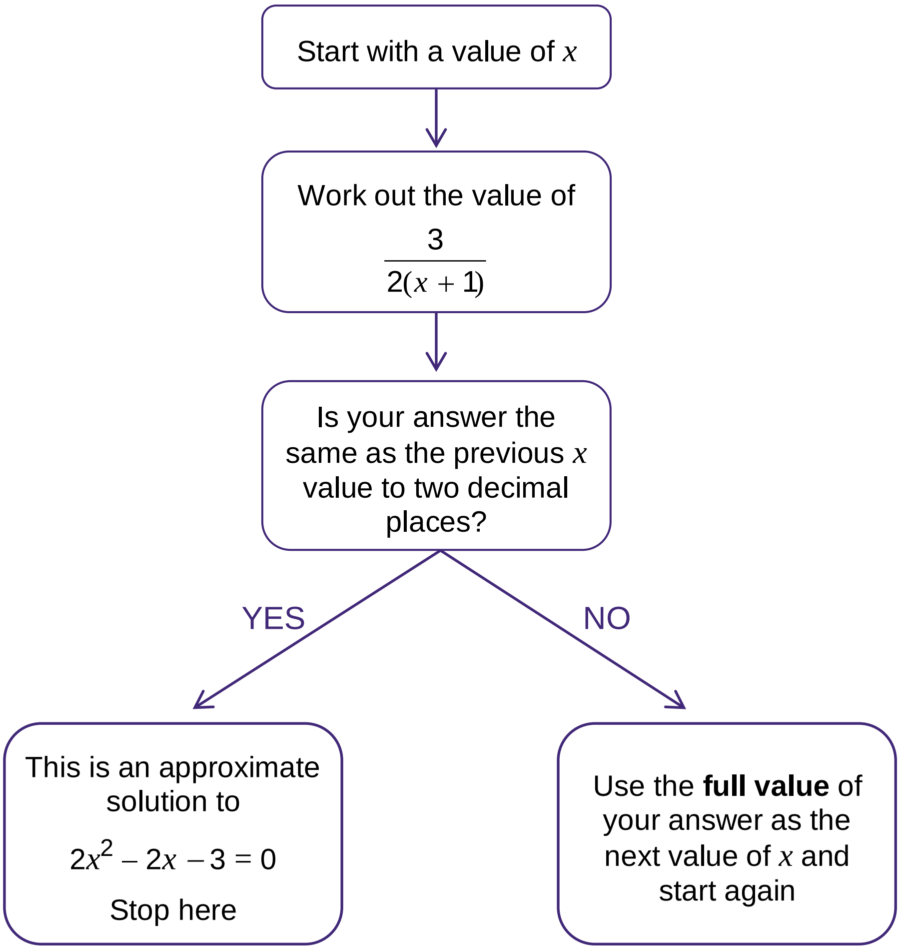
(i) Starting with $x = 1$, use the flow chart to find a solution to the equation $x^3 - 5x + 2 = 0$

Check your answer by substituting it into the equation.

(ii) Repeat (i) starting with $x = -2$.

(iii) What happens when you repeat the process starting with $= -3$?

5 The flow chart is to be used to find solutions to the equation $2x^2 - 2x - 3 = 0$



(i) Starting with $x = 1$, use the flow chart to find a solution to the equation $2x^2 - 2x - 3 = 0$

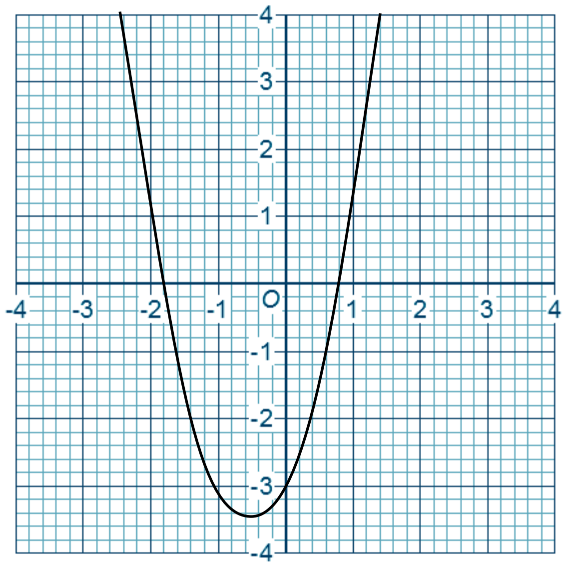
Check your answer by substituting it into the equation.

(ii) The flow chart is now amended so that the formula in the second box is $-\sqrt{\dfrac{3-2x}{2}}$

Starting with $x = 0$, use the flow chart to find a second solution to the equation $2x^2 - 2x - 3 = 0$ Check your answer by substituting it into the equation.

(iii) The graph of $y = 2x^2 + 2x - 3$ is pictured (right)

How do the solutions you found in parts (i) and (ii) relate to the graph?



Skills Builder 3: Iterative notation and finding an iterative formula

1 Javed is solving a problem and needs to know the solutions of the equation $5x^2 - 3x - 4 = 0$

He plans to use an iterative method.

(i) Which **two** of the following are suitable rearrangements of $5x^2 - 3x - 4 = 0$ to use in an iterative formula?

$$x = \frac{5x^2 - 4}{3} \qquad x^2 = \frac{3x + 4}{5} \qquad x = \frac{3x + 4}{5x} \qquad x = \frac{4 + 5x^2}{3}$$

(ii) Javed decides to use the iterative formula $u_{n+1} = \sqrt{\frac{3u_n + 4}{5}}$ starting with initial value $u_1 = 5$

Find a solution to Javed's equation to 2 decimal places.

(iii) He now uses the iterative formula $u_{n+1} = \sqrt{\frac{3u_n + 4}{5}}$ starting with initial value $u_1 = 0$

Find a second solution to Javed's equation to 2 decimal places.

(iv) Check your answers are correct by substituting the solutions back into the original equation.

2 Match the following equations to the corresponding iterative formula.

Complete the empty boxes with the correct equation or iterative formula.

Equation	Iterative formula
$3x^3 - 5x + 1 = 0$	$u_{n+1} = \frac{1-5u_n}{u_n^2}$
$5x^3 - x^2 - 1 = 0$	$u_{n+1} = u_n^2(un + 5)$
$5x^3 - 3x^2 - x = 0$	
	$u_{n+1} = \frac{1-5u_n^3}{u_n}$
$3x^3 + 3x^2 - 5 = 0$	$u_{n+1} = \sqrt{\frac{5-5u_n^3}{3}}$
$x^3 + 5x^2 - x = 0$	$u_{n+1} = u_n^2(5u_n - 3)$

3 A problem is being solved which requires the solution to the equation $x^4 + 3x^3 + 2x - 8 = 0$

(i) Show that the equation can be rearranged to produce the iterative formula

$$u_{n+1} = \sqrt[3]{\frac{8-2u_n - u_n^4}{3}}$$

(ii) Use the iterative formula to carry out eight iterations, starting with the value $u_1 = 4$
Comment on your answer.

(iii) Use the iterative formula to find a solution to the equation to 2 decimal places, starting with $x = 0$

Comment on the pattern of values from each iteration in relation to the final solution.

Check your solution is correct by substituting the final value back into the original equation.

4 A problem is being solved which requires the solution to the equation $x^4 - 2x^3 + 4x^2 - 5x + 7 = 0$

By arranging this equation in different ways, find ten different iterative formulae that could be used to solve the equation

Problem solving 1: Setting up problems in real life contexts

For each of the real life problems below:

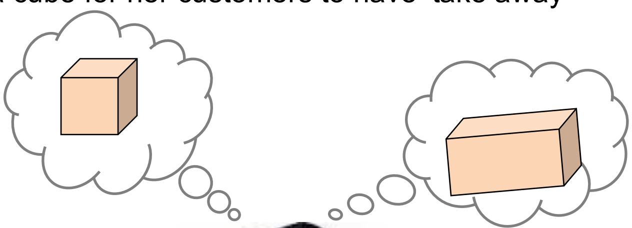
- use algebra to form an equation that represents the problem;
- find the two integer values that are closest to the solution(s);
- use a trial and improvement method to find the solution(s) to one decimal place.

1 Keira is a chef.

She wants to design a strong box in the shape of a cube for her customers to have 'take away' soup.

Each box needs to hold 500 ml of soup.

What should the side length of the cube be?



2 Keira later decides to use a box in the shape of a cuboid.

The cuboid will have a square cross-section and the height will be 10 cm longer than the side length of the square.

The box again needs to hold 500 ml of soup.

What should the side lengths of the box be?

3 A manufacturer makes globes of the earth to be used in classrooms.

They model the globes as spheres.

One particular globe is to be designed to have a volume of 4000 cm^3

Find the radius of the sphere.

Note: Volume of a sphere $= \frac{4}{3}\pi r^2$ where r is the radius.

4 A farmer is building a trough for water for his pigs from metal.

The trough is a prism and the cross-section is a trapezium, as shown.

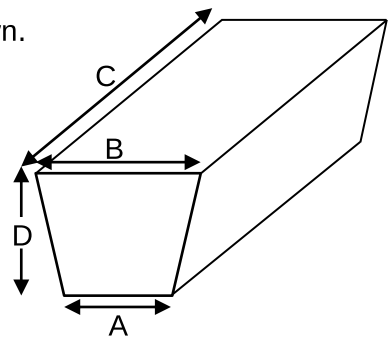
The trough must hold 100 litres.

The side labelled B is 15cm longer than the side labelled A.

Side C is 110cm longer than side A.

Distance D is 5cm longer than side A.

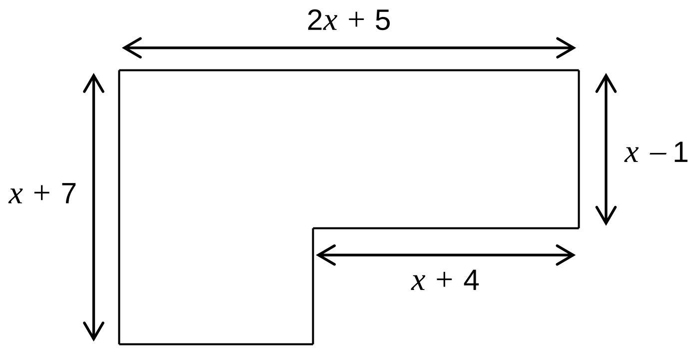
Find the dimensions of the trough.



Did you know that
1 litre = 1000 ml
and each ml has a volume
of 1 cm^3 ?

Problem solving 2: Iterative methods to solve real life problems

1 The diagram below shows the floor layout of a new office that is to be built. To accommodate the required number of staff, the floor space area must be 500 m^2.



(i) Show that the floor space area is given by the equation $2x^2 + 11x - 497 = 0$

(ii) By rearranging the equation in (i) show that two iterative formulae that can be used to solve the problem are $u_{n+1} = \sqrt{\frac{497 - 11u_n}{2}}$ and $u_{n+1} = \frac{497 - 11u_n}{2u_n}$

(iii) By choosing appropriate initial values for u_1 show that there are two possible solutions to the problem, giving your answers to one decimal place.

Explain why one of the solutions is not appropriate.

2 A scientist is measuring tidal patterns and records the height of the sea above or below a fixed point on the harbour wall. During a particular period, the height (in metres) can be modelled by the function $x^3 - 5x^2 + 3x - 2$ where x represents the time (in hours) after the tidal patterns started to be measured.

(i) The scientist wants to find the time at which the height of the tide is 3 metres above the point on the harbour wall. Show that a suitable iterative formula would be $u_{n+1} = \sqrt[3]{5u_n^2 - 3u_n + 5}$ and use this with $u_n = 0$ to find to one decimal place the required time.

(ii) The scientist also wants to find the time at which the height of the tide is 2 metres below the point on the harbour wall.

Show that a suitable iterative formula would be $u_{n+1} = \sqrt{\frac{u_n(u_n^2 + 3)}{5}}$ and use this with $u_1 = 1$ to find to two decimal places the required time.

3 A chemist is studying the rate of cooling of a substance. She has found that the temperature in degrees Celsius (T) at a given point in time (t minutes) is given by the formula $T = \frac{T_0}{5 + t - t^2}$ where T_0 is the initial temperature of the substance.

(i) If the initial temperature of the substance is 50°C, and the chemist wants to know the time taken to reach 20°C, show that a possible iterative formula that could be used to solve the problem is $u_{n+1} = \sqrt{u_n + 2.5}$

Use this iterative formula with $u_1 = 3$ to find the time, in minutes, taken to reach a temperature of 20°C.

Give your answer to two decimal places.

(ii) Suppose the initial temperature of the substance is 80°C, and the chemist again wants to know the time taken to reach 20°C.

Show that a possible iterative formula that could be used to solve the problem is $u_{n+1} = \sqrt{1 + u_n}$

Use this iterative formula with $u_1 = 3$ to find the time, in minutes, taken to reach a temperature of 20°C.

Give your answer to two decimal places.

(iii) The chemist thinks the formula for the cooling of the substance might not be correct.

Using the answers to parts (i) and (ii), do you agree?

4 A hot air balloon manufacturer is designing a balloon that must have a capacity of 2500 m^3.

To find a rough approximation as to the size of the balloon, the manufacturer decides to model the balloon as a sphere.

The volume of a sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3 \qquad \text{where } r \text{ is the radius.}$$

Show that a suitable iterative formula to solve this problem is

$$u_{n+1} = \sqrt{\frac{1875}{\pi u_n}}$$

Using a starting value of $u_1 = 10$ find the radius, in metres, needed to produce a volume of 2500 m^3, giving your answer to two decimal places

Answers

Skills builder 1: Formulating problems and finding approximate integer solutions

1 $3x^2 = 800$

Nearest integers are 16 and 17

2 $\frac{x^3}{6} = 40$

Nearest integers are 6 and 7

3 (a) $(x + 3)^2 + x$

(b) Nearest integers are 17 and 18

4

Problem	Equation	Nearest integers to the solution
Square the number and add the original number to get an answer of 500	$x^3 + 2x = 500$	19 and 20
Cube the number and add double the original number to get an answer of 500	$(x + 2)^2 + x = 500$	21 and 22
Square the number and add double the original number to get an answer of 500	$x^3 - 3x = 500$	21 and 22
Add two to the number and square it then add the original number to get an answer of 500	$x^2 + x = 500$	8 and 9
Cube the number and subtract three times the original number to get an answer of 500	$x^2 + 2x = 500$	7 and 8

Skills builder 2: Basic trial and improvement and iterative flow charts

1 Relevant values are

Trial value	Cube of trial value	Comment
4	64	Too low
5	125	Too high
4.5	91.125	Too low
4.6	97.336	Too low
4.7	103.823	Too high
4.65	100.544625	Too high

Midpoint is too high and so solution is below 4.65 and must therefore round to 4.6 to one decimal place, (or just a comment that 4.6 gives the nearest value to 100, if not checking mid-points).

2 (i) $x^2 + x = 60$

(ii) Relevant values are

Trial value	$x^2 + x$	Comment
7	56	Too low
8	72	Too high
7.5	63.75	Too high
7.4	62.16	Too high
7.3	60.59	Too high
7.2	59.04	Too low
7.25	59.8125	Too low

Midpoint is too low and so the solution is above 7.25 and must therefore round to 7.3 to one decimal place (or just a comment that 7.3 gives the nearest value to 60, if not checking mid-points).

3 (i) $x^3 + 2x = 100$

(ii) Relevant values are

Trial value	$x^3 + 2x$	Comment
4	72	Too low
5	135	Too high
4.5	100.125	Too high
4.4	93.984	Too low
4.45	97.021125	Too low

Midpoint is too low and so the solution is above 4.45 and must therefore round to 4.5 to one decimal place, (or just a comment that 4.5 gives the nearest value to 100, if not checking mid-points).

4 (i) Relevant values are

Starting with $x = 1$
1
0.4432
0.4174112219
0.4145452891
0.4142477389

The last two values both round to 0.41 to 2 d.p.

Check the solution $x = 0.41$: $0.41^3 – (5 \times 0.41) + 2 = 0.018921$ which is very close to zero.

(ii) Relevant values are

Starting with $x = –2$
–2
–1.2
0.0544
0.4000321978
0.4128

The last two values both round to 0.41 to 2 d.p

We do not need to check as $x = 0.41$ is the same solution as in part (i)

5 (i) Relevant values are

Starting with $x = 1$
1
0.75
0.857142857…
0.8076923
0.82978723404
0.81976744186
0.8242811502

The last two values both round to 0.82 to 2 d.p

Checking the answer: $2(0.8242811502)^2 + 2(0.8242811502) - 3 = 0.00744112955$ which is very close to zero.

(ii) Relevant values are

Starting with $x = 0$
0
–1.5
–1.732050808
–1.797790535
–1.815981975
–1.820983793

The last two values both round to –1.82 to 2 d.p

Checking the answer: $2(-1.820983793)^2 + 2(-1.820983793) - 3 = -0.01000363726$ which is very close to zero.

(iii) The graph is of the quadratic function $y = 2x^2 + 2x - 3$ and the solutions obtained in part (i) and (ii) can be seen where the graph cuts the x-axis.

Skills builder 3: Iterative notation and finding an iterative formula

1 (i) The first and third rearrangements are suitable. The second is not suitable because it starts with x^2 rather than x; the fourth is not suitable as it has been rearranged incorrectly (sign error)

(ii)

Starting with x = 5
5
1.949358869
1.403429842
1.281428073
1.252540156
1.2456019
1.243929717
1.243526369

One solution is 1.24 to 2 d.p.

The last two values both round to 1.24 to 2 d.p

(ii)

Starting with x = 0
0
–0.894427191
–0.5131702304
–0.7014968722
–0.6157124951
–0.6561802366
–0.6374102745
––0.6461840568
–0.6420977853
–0.6440041373

The second solution is –0.64 to 2 d.p.

The last two values both round to –0.64 to 2 d.p

(iv) Substituting the full values for each solution into the calculator gives 0.001210044…. and 0.00571905…. respectively, which are both approximately zero. Hence the solutions are correct.

2

Equation	Iterative formula
$3x^3 - 5x + 1 + 0$	$u_{n+1} = \frac{1 - 5u_n}{u_n^2}$
$5x^3 - x^2 - 1 + 0$	$u_{n+1} = u_n^2(un + 5)$
$5x^3 - 3x^2 - x = 0$	$u_{n+1} = \sqrt[3]{\frac{5u_n - 1}{3}}$
$x^3 + 5x - 1 = 0$	$u_{n+1} = \frac{1 - 5u_n^3}{u_n}$
$3x^3 + 3x^2 - 5 = 0$	$u_{n+1} = \sqrt{\frac{5 - 3u_n^3}{3}}$
$x^3 + 5x^2 - x = 0$	$u_{n+1} = u_n^2(5un - 3)$

Note that the missing iterative formula could have a different format to that shown above, depending upon how the student rearranged the formula.

3

(i) Correct rearrangement shown.

(ii) The iterative formula produces values which diverge when starting with $x = 4$

Starting with $x = 4$
4
–4.402569665
–4.927307414
–5.754217682
–7.106822521
–9.446278455
–13.8298502
–23.00956279
–45.37425454

(iii)

Starting with $x = 0$
0
1.386722549
0.7987181043
1.25961179
0.9959175214
1.187556217
1.066186123
1.151074041
1.095324747
1.133581739
1.108075745
1.125420634
1.113780283
1.121662692
1.116357086

The last two values both round to 1.12 to 2 d.p.

The answers to each iteration oscillate to the solution i.e. one above, one below, one above, etc… Checking the final value in the equation gives a value of -0.04034… which is very close to zero.

4 When students have found as many formulae as they can, their results can be checked either:

- by a partner comparing their answers to see if any match, and/or
- by each student rearranging their partner's equations to ensure they get back to the original equation.

Problem Solving 1: Setting up problem in real life contexts

1 $x^3 = 500$

Integer values are 7 and 8

Relevant values are

Trial value	x^3	Comment
7	343	Too low
8	512	Too high
7.9	493.039	Too low
7.95 (if checking midpoint)	502.459875	Too high

7.9 is correct to one decimal place

2 $x^2 (x + 10) = 500$

Integer values are 5 and 6

Relevant values are

Trial value	$x^3(x + 10)$	Comment
5	375	Too low
6	576	Too high
5.5	468.875	Too low
5.6	489.216	Too low
5.7	510.093	Too high
5.65 (if checking midpoint)	499.587125	Too low

5.7 is correct to one decimal place

3 $\frac{4}{3}\pi r^3 = 4000$

Integer values are 9 and 10

Relevant values are

Trial value	$\frac{4}{3}\pi r^3$	Comment
9	3053.628059	Too low
10	4188.790205	Too high
9.5	3591.364002	Too low
9.7	3822.995723	Too low
9.8	3942.45583	Too low
9.9	4064.378947	Too high
9.85 (if checking midpoint)	4003.107942	Too high

5.7 is correct to one decimal place

4 100 litres = 100000 ml = 100000 cm^3

Area of trapezium = $\frac{1}{2}(a + b)h = \frac{1}{2}(A + A + 15)(A + 5) = \frac{1}{2}(2A + 15)(A + 5)$

So volume of trough = $\frac{1}{2}(2A + 15)(A + 5) \times (A + 110) = 100000$

Integer values for *A* are 21 and 22

Trial value	$\frac{1}{2}(2A + 15)(A + 5)(A + 110)$	Comment
21	97071	Too low
22	105138	Too high
21.5	101057.75	Too high
21.4	100252.944	Too high
21.3	99.451.872	Too low
21.35 (if checking midpoint)	99851.94163	Too low

So *A* = 21 to one decimal place. Therefore the dimensions of the trough (in centimetres) are *A* = 21.4, *B* = 36.4, *C* = 131.4 and *D* = 26.4

Problem Solving 2: Iterative methods to solve real life problems

1 (i) For example

$(2x + 5)(x + 7) – 8(x + 4) = 500$

$2x^2 + 19x + 35 – 8x – 32 = 500$

$2x^2 + 11x – 497 = 0$

(ii) Correct rearrangements shown

(iii) Using $u_{n+1} = \frac{497 - 11u_n}{2u_n}$ with $u_1 = 5$ (say) gives the following values

Starting with $u_1 = 5$
5
14.86606875
12.91265356
13.3221772
13.2373723
13.2549784
13.25132517

The last two values both round to 13.3 to 1 d.p.

Using $u_{n+1} = \frac{497 - 11u_n}{2u_n}$ with $u_1 = -2$ (say) gives the following values

Starting with $x = -2$
–2
–129.75
–7.415221579961
–39.01214759
–11.86981082
–26.43546424
–14.90025103
–22.17757137
–16.70501411
–20.37577313
–17.69585625
–19.5428356
–18.21565729
–19.14210997
–18.48184998
–18.94562369
–18.61648558
–18.84838409
–18.68415408
–18.80004018
–18.71805686
–18.7759507
–18.73501558
–18.76393346
–18.74349186
–18.75793517
–18.74772677
–18.75494035
–18.7498422
–18.75344488
–18.75089879

The last two values both round to –18.8 to 1 d.p.

The second value is not appropriate as it is negative and side lengths cannot be negative.

2 (i) $x^3 - 5x^2 + 3x - 2 = 3$

$x^3 = 5x^2 - 3x + 5$

$x^3 = \sqrt[3]{5x^2 - 3x + 5}$

$u_{n+1} = \sqrt[3]{5u_n^2 - 3u_n + 5}$

Relevant values are

Starting with $u_1 = 0$
0
1.709975947
2.437947798
3.014892759
3.459484187
3.790504957
4.030354983
4.200750839
4.32013784
4.402985943
4.460099407
4.499294299

The last two values both round to 4.5 to 1 d.p.

So the required time is 4.5 hours

(ii) $x^3 - 5x^2 + 3x - 2 = -2$

$x^3 - 5x^2 + 3x = 0$

$x^3 - 3x = 5x^2$

$$x = \sqrt{\frac{x(x^2 + 3)}{5}}$$

$$u_{n+1} = \sqrt{\frac{u_n(u_n^2 + 3)}{5}}$$

Relevant values are

Starting with $u_1 = 1$
1
0.894427191
0.8244784201
0.7789591205
0.7496041732
0.7307557614
0.7186784777
0.7109475327
0.706001199

The last two values both round to 0.17 to 2 d.p.

So the time taken is 0.71 hours

3 (i) $20 = \dfrac{50}{t + t - t^2}$

$5 + t - t^2 = 2.5$

$t + 2.5 = t^2$

$\sqrt{t + 2.5} = t$

$u_{n+1} = \sqrt{u_n + 2.5}$

Relevant values are

Starting with $u_1 = 3$
3
2.34520788
2.201183291
2.168221227
2.16060668
2.15884383

The last two values both round to 2.16 to 2 d.p.

So the time taken is 2.16 minutes

(ii) $20 = \dfrac{80}{5 + t + t^2}$

$5 + t - t^2 = 4$

$1 + t = t^2$

$\sqrt{1 + t} = t$

$u_{n+1} = \sqrt{1 + u_n}$

Relevant values are

Starting with $u_1 = 3$
3
2
1.732050808
1.65289165
1.628769981
1.621348198
1.619057812

The last two values both round to 1.62 to 2 d.p.

So the time taken is 1.62 minutes to 2 decimal places

(iii) So the model used seems to be incorrect - it takes less time for the substance to cool from 80°C to 20°C than it does to cool from 50°C to 20°C

4

$2500 = \frac{4}{3}\pi r^2$

$7500 = 4\ \pi r^3$

$1875 = \pi r^3$

$\frac{1875}{\pi} = r^3$

$\frac{1875}{\pi r} = r^2$

$\sqrt{\frac{1875}{\pi r}} = r$

$u_{n+1} = \sqrt{\frac{1875}{\pi u_n}}$

Relevant values are

Starting with u_1 = 10
10
7.72548404
8.789473272
8.240323749
8.510470336
8.374307782
8.442114544
8.40814281
8.425111554
8.416622906
8.420866161

The last two values both round to 8.42 to 2 d.p.

So the radius of the balloon is 8.42 m to two decimal places